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ADJUSTMENT PROCESSES AND RADIATING SOLITARY WAVES IN A REGULARISED OSTROVSKY EQUATION

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Abstract – The Ostrovsky equation is an adaptation of the Korteweg-de Vries equation widely used to describe the effect of rotation on surface and internal solitary waves. It has been shown that the effect of rotation is to destroy such solitary waves in finite time due to the emission of trailing radiation. Here this issue is re-examined for a regularised Ostrovsky equation. The regularisation is necessary to remove an anomaly in the Ostrovsky equation whereby there is a discontinuity in the mass field at the initial moment. It is demonstrated that in the regularised Ostrovsky equation there is a rapid adjustment of the mass which is transported a large distance in the opposite direction to that in which the solitary wave propagates. © Elsevier, Paris

1. Introduction

Internal solitary waves are an ubiquitous feature of the ocean (and are also commonly occurring in the atmosphere). In the oceanic environment, they are generally generated by the interaction of currents, often of tidal origin, with topography, and are readily observed through satellite remote sensing techniques. Although they are internal to the ocean, their surface currents are usually sufficiently strong to significantly modulate the wind-wave field.

Recently there has been some interest in the effect of the earth's rotation on these waves (see the review by Grimshaw et. al, [1]). In the simplest approximation, the effects of rotation are described by the so-called Ostrovsky equation [2], presented here in dimensionless form

$$u_t + 3uu_x + \frac{1}{4}u_{xxx} = \epsilon^2 v\,, \tag{1}$$

$$v_x = u\,. \tag{2}$$

The parameter ϵ represents the effects of rotation. Although this is a planar equation, the rotational terms are a manifestation of the Coriolis force which of course introduces a three-dimensional aspect to the evolution of the waves. When $\epsilon = 0$, equation (1) reduces to the familiar KdV equation which supports the well-known solitary wave solution. But for $\epsilon > 0$, it can be shown that there are no exact steady solitary wave solutions [3, 4]. Instead, if the KdV solitary wave is used as an initial condition for (1), it will decay to zero in finite time due to the emission of trailing radiation [5].

One of the interesting features of the Ostrovsky equation (1) and (2), is that for any localised solution (i.e. $u \to 0$ as $|x| \to \infty$) the mass is zero for any $t > 0$, that is

$$M = \int_{-\infty}^{\infty} u\,dx = 0 \tag{3}$$

Indeed, if $u \to 0$ as $|x| \to \infty$, then (1) shows that $v \to 0$ as $|x| \to \infty$ also, and then the result (3) follows from (2). This is the case, even when the initial condition,

$$u = u_0(x) \quad \text{at} \quad t = 0 \tag{4}$$

is such that the initial mass

$$M_0 = \int_{-\infty}^{\infty} u_0 \, dx \tag{5}$$

is not zero. The physical explanation is that just at the initial moment, an infinitely long wave of zero height, but finite mass, propagates to negative infinity with infinite speed (see the discussion in Grimshaw et. al 1998b for instance).

Our main aim in this paper is to examine this anomalous situation in more detail by introducing a regularising parameter δ , so that equation (2) is replaced by

$$-\delta^2 v_t + v_x = u\,. \tag{6}$$

The physical motivation for this is discussed by Grimshaw and Melville [6]. Recently Ablowitz and Wang [7] introduced a related regularisation for the Kadmotsev-Petviashvili equation, where a similar issue regarding the mass arises.

Briefly, in physical terms, equation (2) describes the momentum balance in the direction transverse to the wave propagation direction (i.e. the x-direction). In essence, u can be regarded as the fluid velocity in the x-direction, which in the weakly nonlinear long wave approximation, is equivalent to the vertical wave displacement at leading order, while v is the fluid velocity in the transverse direction. In equation (2) the acceleration term is calculated in the frame of reference moving with the linear long wave speed, and so is represented only by $-v_x$, whereas in equation (6) this same acceleration term retains an additional time evolution with respect to this frame of reference, and so is represented by $\delta^2 v_t - v_x$, where δ is a small parameter. It is now clearly necessary to specify an additional initial condition

$$v = v_0(x) \quad \text{at} \quad t = 0 \tag{7}$$

for the regularised Ostrovsky equations (1) and (6). In Sections 2 and 3 of this paper we analyse the effect of this regularisation in the limit as $\delta \to 0$. Our asymptotic analysis is somewhat similar to that of Ablowitz and Wang [7]. Because the essential aspects of the issue relating to the mass occur also in the linearised long wave version of (1) and (6), these are considered first in Section 2, where the exact solution for the initial conditions (4) and (7) can readily be constructed. Then in Section 3, we consider the full system (1) and (6) and use a boundary-layer analysis to demonstrate that the mass is transported to large negative x on a timescale of $\mathcal{O}(\delta/\epsilon)$, as $\delta \to 0$.

Before proceeding, to these detailed analyses, we note that it can be readily shown from (1) and (6) that

$$M_{tt} + \frac{\epsilon^2}{\delta^2} M = 0\,. \tag{8}$$

Thus the mass M oscillates at the frequency ϵ/δ , which we can identify as the inertial frequency in physical terms. The solution of (8) is clearly

$$M = M_0 \cos(\epsilon t/\delta) + \epsilon\, \delta\, M_1 \sin(\epsilon t/\delta)\,, \tag{9}$$

where $M_1 = \int_{-\infty}^{\infty} v_0(x)\, dx$. Note that M has no formal limit as $\delta \to 0$.

In Section 4 we return to the issue of the fate of a KdV solitary wave, when used as an initial condition for the regularised Ostrovsky equation (1) and (6). We show that to a large extent, the analysis of Grimshaw et.

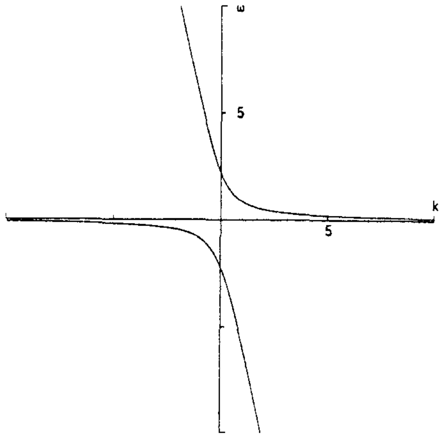

FIGURE 1. The dispersion relation (12) for $\epsilon^2 = 0.5$ and $\delta^2 = 0.1$.

al, [5] for the case $\delta = 0$ carries over here when $\delta \neq 0$, and the wave again decays to zero in finite time due to the emission of trailing radiation. We also note here that there can be no steady solitary wave solution of the regularised Ostrovsky equation (1) and (6). Indeed, if we were to seek such a solution with speed c , the system (1) and (6) would reduce to (1) and (2) with ϵ^2 replaced by $\epsilon^2/1 + \delta^2 c$. Thus the arguments of Leonov [3] or Galkin and Stepanyants [4] can again be used to show there is no such solution.

2. Linearised Equations

The linearised long wave version of the regularised Ostrovsky equation (1) and (6) is obtained by omitting the nonlinear term and the third dispersive term from (1) to obtain

$$u_t = \epsilon^2 v\,, \tag{10}$$

$$-\delta^2 v_t + v_x = u\,. \tag{11}$$

These are to be solved with the same initial conditions, namely (4) and (7). Note that this reduced system has exactly the same solution (9) for the mass M as the full system.

First, we note that the dispersion relation for waves of wavenumber k and frequency ω is

$$\delta^2\omega^2 + \omega k = \epsilon^2\,. \tag{12}$$

The two branches are plotted in Figure 1. When $\delta = 0$, there is just a single branch with anomalous behaviour as $k \to 0$. In this limit $\delta = 0$ infinitely long wave have infinitely large negative group velocity. But when $\delta \neq 0$, both branches have group velocity $-1/2\delta^2$ at $k = 0$, which is large but finite when $0 < \delta << 1$, while the corresponding frequencies are $\pm\epsilon/\delta$. The two branches are given by

$$\omega = \frac{\epsilon\,\Omega_\pm(K)}{\delta}\,, \tag{13}$$

where

$$k = \epsilon\,\delta\,K\,, \tag{14}$$

and

$$\Omega_{\pm}(K) = -\frac{1}{2}K \pm \left(1 + \frac{1}{4}K^2\right)^{1/2} . \tag{15}$$

Note that $\Omega_+(K) = -\Omega_-(K)$.
The solution of the initial value problem for equations (10) and (11) is

$$u = \int_{-\infty}^{\infty} a_+(k)\, e^{ikx - i\Omega_+(K)T}\, dk + \int_{-\infty}^{\infty} a_-(k)\, e^{ikx - i\Omega_-(K)T}\, dk\,, \tag{16}$$

$$v = \int_{-\infty}^{\infty} \left[-\frac{i}{\epsilon\delta}\Omega_+(K)\, a_+(k)\right] e^{ikx - i\Omega_+(K)T}\, dk + \int_{-\infty}^{\infty} \left[-\frac{i}{\epsilon\delta}\Omega_-(K)\, a_-(k)\right] e^{ikx - i\Omega_-(K)T}\, dk\,, \tag{17}$$

where

$$T = \frac{\epsilon\, t}{\delta} . \tag{18}$$

The amplitudes $a_{\pm}(k)$ are determined from the initial conditions. It is readily shown that

$$a_{\pm}(k) = \frac{\epsilon\,\delta\,\hat{v}_0(k) + i\,\Omega_{\mp}(K)\,\hat{u}_0(k)}{\mp i\,[\Omega_+(K) - \Omega_-(K)]}\,, \tag{19}$$

where

$$(\hat{u}_0(k), \hat{v}_0(k)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (u_0(k), v_0(k))\, \exp(-ikx)\, dx\,. \tag{20}$$

Because $(u_0(k), v_0(k))$ are real-valued, it is readily shown that $\overline{a_+(k)} = a_-(k)$.

Next, we consider this solution in the limit when the regularising parameter $\delta \to 0$, while the rotational parameter ϵ remains fixed, with $\epsilon > 0$. First, we note that the temporal development is completely defined in terms of the variable T (18), and so the solution develops on the short timescale where t is $\mathcal{O}(\delta/\epsilon)$. Hence the limit $\delta \to 0$ can be replaced here by $T \to \infty$. This limit can now be readily obtained using the method of stationary phase. It is not necessary here to present all details, but we note the salient points. First, the stationary phase condition can be expressed as

$$\left.\begin{aligned} \frac{X}{T} &= \Omega_{\pm}{}'(K)\,, \\ \text{where}\quad X &= \epsilon\,\delta\, x\,. \end{aligned}\right\} \tag{21}$$

Since the group velocity $\Omega_{\pm}{}'(K)/\delta^2$ is large and negative, each of these branches represents a wave which has propagated to large negative x. Next we note that in this limit u is $\mathcal{O}(\delta)$, while v is $\mathcal{O}(1)$. The amplitudes are given by (19) where as $\delta \to 0$,

$$(\hat{u}_0, \hat{v}_0) \to \frac{1}{2\pi}(M_0, M_1)\,. \tag{22}$$

Here we recall that (M_0, M_1) are defined by (5) and (9), and represent the initial mass in (u, v) . Thus these waves are responsible for transporting the mass to large negative x.

Of particular interest is the nature of the solution in the region where x remains of $\mathcal{O}(1)$ while $T \to \infty$. From the stationary phase condition (21), this requires $\Omega_{\pm}{}'(K)$ to be $\mathcal{O}(\epsilon\,\delta)$, and then from the dispersion relation

(15) it follows that $K \to \infty$ for the first branch, while $K \to -\infty$ for the second branch. Thus, as $K \to \infty$, $\Omega_+(K) \sim K^{-1}$, $\Omega_+'(K) \sim -K^{-2}$ and so K is $\mathcal{O}(1/\sqrt{\epsilon\,\delta})$, while $a_+(k) \sim \hat{u}_0(k)$. Similarly, as $K \to -\infty$, $\Omega_-(K) \sim K^{-1}$, $\Omega_-'(K) \sim -K^{-2}$, and $a_-(k) \sim \hat{u}_0(k)$. In these same limits, $a_\pm(k)$ are $\mathcal{O}(\delta)$ respectively. It follows that,

$$u \sim \mathcal{P.V.} \int_{-\infty}^{\infty} \hat{u}_0(k) \exp(ikx - i\epsilon^2 t/k)\, dk\,, \tag{23}$$

where $\mathcal{P.V.} \int_{-\infty}^{\infty}$ denotes the principal value. But the right-hand side of (23) is precisely the exact solution of equations (10) and (11) when $\delta = 0$, which satisfies the initial condition (4). The corresponding solution for v is obtained from (23) by replacing $\hat{u}_0(k)$ with $\hat{u}_0(k)/ik$. It can be shown that the asymptotic solution (23) satisfies the zero mass condition (3).

In summary, we can conclude that for the system (10) and (11), there is a temporal boundary layer of $\mathcal{O}(\delta)$, which propagates in the negative x-direction a distance of $\mathcal{O}(1/\epsilon\delta)$. This wave is responsible for the mass M, and hence for the expression (9). The remaining part of the solution remains in the region where x is $\mathcal{O}(1)$, and to within an error of $\mathcal{O}(\delta)$, satisfies the reduced system (10) and (11) with $\delta = 0$.

3. Matching to the Temporal Boundary Layer

Here we shall consider the full regularised Ostrovsky equations (1) and (6), with the initial conditions (4) and (7), in the limit as the regularising parameter $\delta \to 0$, with $\epsilon > 0$ held fixed. It is clear from the analysis of the linearised long wave equations in the previous section that there is a temporal boundary layer of $\mathcal{O}(\delta/\epsilon)$ just after the initial time $t = 0$. Further, the spatial domain at this time can be broken down into two segments, a near-field segment where x is $\mathcal{O}(1)$, and a far-field segment where x is $\mathcal{O}(1/\epsilon\delta)$ and is negative.

To describe the second segment of this temporal boundary layer, we introduce the scaled variables $T = \epsilon t/\delta$ (18) and $X = \epsilon\delta x$ (21), and put

$$u = \delta\, U\,, \quad V = \epsilon\, v\,. \tag{24}$$

With this scaling, the equations (1) and (6) become

$$\begin{aligned} U_T + 3\,\delta^2\, U U_x + \frac{1}{4}\,\epsilon^2\,\delta^3\, U_{xxx} &= V\,, & (25)\\ -V_T + V_x &= U\,. & (26)\end{aligned}$$

The initial conditions (4) and (7) become

$$\begin{aligned} U &= U_0(X) = \frac{1}{\delta}\, u_0\left(\frac{X}{\epsilon\,\delta}\right), \quad \text{at} \quad T = 0\,, & (27)\\ V &= \epsilon\, v_0\left(\frac{X}{\epsilon\,\delta}\right), \quad \text{at} \quad T = 0\,. & (28)\end{aligned}$$

Taking the limit as $\delta \to 0$, we now see that to within an error of $\mathcal{O}(\delta^2)$, equations (25) and (26) reduce to

$$\left.\begin{aligned} U_T &= V\,,\\ -V_T + V_X &= U\,,\end{aligned}\right\} \tag{29}$$

while the initial conditions become

$$\left.\begin{aligned} U &= \epsilon\, M_0\,\hat{\delta}(X)\,, \quad \text{at} \quad T = 0\,,\\ V &= \delta\,\epsilon^2\, M_1\,\hat{\delta}(X)\,, \quad \text{at} \quad T = 0\,.\end{aligned}\right\} \tag{30}$$

Here we let $\hat{\delta}(.)$ denote the Dirac δ-function. We see that equations (29) are just, after rescaling, the linearised long wave equations (10) and (11) discussed in the previous section. But importantly, the initial conditions (4) and (7) are now reduced to (30), consistent with the result (22). Thus the solution of (29) and (30) is, after rescaling, just (16) and (17) with $(\hat{u}_0, \hat{v}_0)$ given by (22). This demonstrates that, just as in the linearised long wave system of §2, the mass is carried to large negative x in a time of $\mathcal{O}(\delta/\epsilon)$ at a speed of $\mathcal{O}(\delta^{-2})$.

In the first segment of the temporal boundary layer, we again rescale the time variable to $T = \epsilon t/\delta$ but now retain x, to get

$$\left.\begin{aligned} U_T + \tfrac{\delta}{\epsilon}\,(3uu_x + \tfrac{1}{4}\,u_{xxx}) &= \epsilon\delta\, v\,, \\ -\,\epsilon\,\delta\, v_T + v_x &= u\,. \end{aligned}\right\} \tag{31}$$

The initial conditions (4) and (5) are unchanged. However, it is immediately apparent that in the limit $\delta \to 0$ this system cannot satisfy the initial conditon (7) for v. Hence, it is necessary to introduce an even finer time scale,

$$\tau = \frac{t}{\delta^2}\,. \tag{32}$$

We now get

$$\left.\begin{aligned} u_\tau + \delta^2\,(3uu_x + \tfrac{1}{4}\,u_{xxx}) &= \epsilon^2\,\delta^2\, v\,, \\ -\,v_\tau + v_x &= u\,. \end{aligned}\right\} \tag{33}$$

With an error of $\mathcal{O}(\delta^2)$, the solution of the initial value problem of the system (33) is

$$\left.\begin{aligned} u &= u_0(x)\,, \\ \text{and} \quad v &= v_0(x+\tau) + \textstyle\int_{x+\tau}^{x} u_0(x')\,dx'\,. \end{aligned}\right\} \tag{34}$$

Thus, on this very short time scale, we see that u is unchanged to $\mathcal{O}(\delta^2)$, while v consists of a steady part together with a wave propagating to large negative x. Indeed, we see that,

$$v \to \int_{\infty}^{x} u_0(x')\,dx'\,, \quad \text{as} \quad \tau \to \infty\,. \tag{35}$$

We can now return to the system (31), whose initial condition as $T \to 0_+$ is just the solution (34) of the system (33) as $\tau \to \infty$. Thus, to $\mathcal{O}(\delta)$, we see that the solution of (31) is just $u = u_0(x)$, with v given by (35).

Next, it is necessary to match the solutions in the first and second segments spatially, on the temporal boundary layer timescale. Thus, in the first segment we observe that as $x \to -\infty$, $u \to 0$, but $v \to -M_0$. Since $u = \delta U$ is $\mathcal{O}(\delta)$ in the second segment, this is consistent for the dependent variable u, while for the dependent variable v it implies that $V \to -\epsilon M_0$ as $X \to 0_-$, at least for $T > 0$ This latter requirement is readily confirmed from the solution of the system (29), with the initial conditions (30), which can be obtained from the analysis of the previous section.

It is now apparent that, within the temporal boundary layer, the mass M (3), is given by

$$\left.\begin{aligned} M &= M_0 + N\,, \\ \text{where} \quad N &= \textstyle\int_{-\infty}^{\infty} U\,dx\,. \end{aligned}\right\} \tag{36}$$

Here M_0 is the mass in the first segment, and N that in the second segment. It is readily shown form (29) and the condition $V \to -\epsilon M_0$ as $X \to 0_-$, that

$$N_{TT} + N = -M_0 \,, \quad T > 0 \,. \tag{37}$$

Of course, the expressions (36) and (37) are completely equivalent to the exact expression (9), and confirm that the time evolution of the mass is due to the waves generated in the second segment. This last comment also implies that, although the above analysis would seem to suggest that u remains time-independent, to $\mathcal{O}(\delta)$, within the temporal boundary layer in the first segment, this cannot hold eventually as $T \to \infty$. Indeed, this is precisely the implication of the solution (23) of the linearised long wave problem in the first segment.

The final step in this section is to consider the near-field segment where x is $\mathcal{O}(1)$, on the time-scale where t is $\mathcal{O}(1)$, and $t > 0$. In the limit when the regularising parameter $\delta \to 0$ with ϵ fixed, it is readily apparent that we obtain the Ostrovsky equations (1) and (2). However, the initial condition is now obtained by matching with the solution in the temporal boundary layer as $T \to \infty$. It is readily seen that the appropriate initial condition is just (4) for u, and no initial condition is required for v. It also now follows that for the system (1) and (2), the zero mass condition holds for all $t > 0$, even although $M_0 \neq 0$ in general.

4. Radiation damping of a solitary wave

Next we re-examine the fate of a KdV solitary wave due to the effect of rotation, considered by Grimshaw et. al [5] for the Ostrovsky equations (1) and (2). Here, we consider the same problem but for the regularised equations (1) and (6). Thus we now assume that $0 < \epsilon^2 << 1$, and construct an asymptotic theory valid as $\epsilon \to 0$, with the regularising parameter δ kept fixed, albeit at a small value. This is in marked contrast to the previous sections, where we kept ϵ fixed, and let $\delta \to 0$. Nevertheless, it turns out that the radiation damping of a solitary wave proceeds very much as in the case $\delta = 0$, and hence we need give only a brief outline here.

The asymptotic solution consists of three parts, an "inner" solution in the solitary wave region, a "near-field" radiation solution which co-propagates with the solitary wave, and a "far-field" radiation field. Thus, the "inner" solution is described in terms of the variables

$$T = \epsilon^2 t \,, \quad \theta = x - \int_0^t c(T')\, dT' \,. \tag{38}$$

We put

$$\left.\begin{array}{rcl} u &=& u^{(0)}(\theta,T) + \epsilon^2 u^{(1)}(\theta,T) + \dots \,, \\ v &=& v^{(0)}(\theta,T) + \epsilon^2 v^{(1)}(\theta,T) + \dots \,, \\ c &=& c^{(0)} + \epsilon^2 c^{(1)} + \dots \end{array}\right\} \tag{39}$$

and readily find that the leading term corresponds to the KdV solitary wave,

$$\begin{array}{rcl} u^{(0)} &=& A\,\mathrm{sech}^2(\sqrt{A}\,\theta)\,, \\ v^{(0)} &=& -\dfrac{\sqrt{A}}{1+\delta^2 A}\left[1 - \tanh(\sqrt{A}\,\theta)\right] \,, \\ c^{(0)} &=& A\,. \end{array} \tag{40}$$

At the next order we obtain a linear equation for $u^{(1)}$,

$$\left.\begin{array}{rcl} -\,c^{(0)}\, u^{(1)}_\theta + 3\,(u^{(0)}\, u^{(1)})_\theta + \frac{1}{4}\, u_{\theta\theta\theta}{}^{(1)} &=& -\, u_T{}^{(0)} + c^{(1)}\, u_\theta{}^{(0)} + v^{(0)}\,, \\ (1+\delta^2 A)\, v_\theta{}^{(0)} - u^{(1)} &=& \delta^2\, c^{(1)}\, v_\theta{}^{(0)} - \delta^2\, v_T{}^{(0)}\,. \end{array}\right\} \tag{41}$$

The compatibility condition for $u^{(1)}$ is that

$$\int_{-\infty}^{\infty} u^{(0)} \left(- u_T{}^{(0)} + v^{(0)} \right) d\theta = 0 \,. \tag{42}$$

Using the expressions in (40) we readily find that (42) becomes

$$\frac{d}{dT} \left(\frac{2}{3} A^{3/2} \right) + \frac{2\,A}{1 + \delta^2\, A} = 0 \,, \tag{43}$$

for which the solution is

$$A^{1/2}\,(1 + \frac{1}{3}\,\delta^2\, A) = T_0 - T \,. \tag{44}$$

Thus, just as in the case $\delta = 0$, the amplitude A decreases to zero in finite time, and is extinguished at $T = T_0$. Note that $T_0 = {A_0}^{1/2}\,(1 + \frac{1}{3}\,\delta^2\, A_0)$ where A_0 is the initial amplitude.

The "near-field" radiation zone is the region immediately behind the solitary wave. First we note that, as $\theta \to -\infty$, $u^{(0)} \to 0$, while

$$u^{(1)} \to \frac{2\,\theta}{\sqrt{A}\,(1 + \delta^2\, A)} \,, \quad \text{and} \quad v^{(0)} \to - \frac{2\,\sqrt{A}}{1 + \delta^2\, A} \,. \tag{45}$$

These provide the matching conditions for the "near-field" radiation, which is described by

$$u = \epsilon\, U(X,T) \,, \quad v = V(X,T) \,, \quad X = \epsilon\, \theta \tag{46}$$

where we recall that T and θ are defined in equation (38). At the lowest order, the "near-field" equations are

$$\left. \begin{aligned} - A\, U_X &= V + v^{(0)} \,, \\ (1 + \delta^2\, A)\, V_X &= U \,. \end{aligned} \right\} \tag{47}$$

Note that although $u^{(0)} \to 0$ in the "near-field" radiation zone, we must retain $v^{(0)}$ in the expression for v. Eliminating V, we get

$$A\, U_{XX} + \frac{U}{1 + \delta^2\, A} = - \frac{1}{\epsilon}\, \frac{u^{(0)}(X/\epsilon)}{1 + \delta^2\, A} \approx - \frac{2\,\sqrt{A}}{1 + \delta^2\, A}\, \hat{\delta}(X) \,. \tag{48}$$

Here $\hat{\delta}(X)$ denotes the Dirac δ-function. The matching condition is $\epsilon\, U(X \to 0) \sim \epsilon^2\, u^{(1)}\,(\theta \to -\infty)$, and from (45) we see that $U \to 2\,X/\sqrt{A}\,(1 + \delta^2\, A)$ as $X \to 0_-$, while of course $U = 0$ for $X > 0$. Thus the solution of equation (48) is, in $X < 0$,

$$U = \frac{2}{\sqrt{1 + \delta^2\, A}} \sin \left[\frac{X}{\sqrt{A\,(1 + \delta^2\, A)}} \right] \,. \tag{49}$$

The corresponding solution for v is, in total, $V + v^{(0)}$, or

$$U = - \frac{2\,\sqrt{A}}{\sqrt{1 + \delta^2\, A}} \cos \left[\frac{X}{\sqrt{A\,(1 + \delta^2\, A)}} \right] \,. \tag{50}$$

Next, the "far-field" radiation zone is unsteady, and best represented in the original variables x and t . Thus with $u = \epsilon U$, $v = V$ as in the "near-field" zone (see (46)), we obtain at the lowest order,

$$\left.\begin{aligned} U_t &= \epsilon V , \\ -\delta^2 V_t + V_x &= \epsilon U . \end{aligned}\right\} \tag{51}$$

This is just the regularised linear long wave equations (10) and (11), and hence have the same dispersion relation (12). The structure of the radiated wave field is readily found using the well-known symptotic theory of slowly-varying wave trains (e.g. Whitham, [8]), and the analysis follows the same path as that described in Grimshaw et. al [5] for the case $\delta = 0$. The solution is thus described by the space-time rays

$$\frac{dx}{dt} = \omega'(k) , \tag{52}$$

where $\omega(k)$ is determined from the dispersion relation (12). On each ray the local frequency ω and wavenumber k remain constant, being determined by their values at the solitary wave trajectory $dx_s/dt = A(t)$. Here, we use the "near-field" solutions to infer that the wavenumber k at the solitary wave trajectory is $\epsilon/\sqrt{A(1+\delta^2 A)}$ (see (49)), from which it follows that the phase speed is A (the solitary wave speed), while the group velocity is $-A/(1+2\delta^2 A)$. Energy conservation determines the wave amplitude in the radiated field, where the wave amplitude at the solitary wave trajectory is found from the "near-field" solution (49) and is $2\epsilon/\sqrt{1+\delta^2 A}$.

Finally, we return to the issue of the mass, being the main focus of this paper. It is readily verified that for this asymptotic solution, the mass M (3) satisfies equation (8). Since the mass of the solitary wave itself is $2\sqrt{A}$, it follows immediately that it is the radiated field which firstly has a term exactly cancelling this mass, and secondly contains the terms required to produce the inertial oscillations in the mass (cf. (9)).

References

[1] R.H.J. Grimshaw, L.A. Ostrovsky, V.I. Shrira and Yu. A. Stepanyants, Nonlinear surface and internal gravity waves in a rotating ocean, Nonlinear Processes in Geophysics 19 (1998) 289-338.

[2] L.A. Ostrovsky, Nonlinear internal waves in a rotating ocean, Okeanologiya, 18 (1978) 181-191 (in Russian); (English translation, Oceanology 18 119-125).

[3] A.I. Leonov, The effect of the earth's rotation on the propagation of weak nonlinear surface and internal long oceanic waves, Ann. NY Acad. Sci., (1981) 373 150-159.

[4] V.M. Galkin and Yu. A. Stepanyants, On the existence of stationary waves in a rotating fluid, Prikl. Mat, Mekl. 55 (1991) 1051-1055 (in Russian); (English translation, J. Appl. Maths. Mech. 55 939-943).

[5] R.H.J. Grimshaw, J-M He and L.A. Ostrovsky, Terminal damping of a solitary wave due to radiation in rotational systems, Stud. Appl. Math. 101 (1998) 197-210.

[6] R. Grimshaw and W.K. Melville, On the derivation of the modified Kadomtsev-Petviashvili equation, Stud. Appl. Math. 80 (1989) 183-202.

[7] M.J. Ablowitz and X-P Wang, Initial time layers and Kadomtsev-Petviashvili-type equations, Stud. Appl. Math. 98 (1997) 121-137.

[8] G.B. Whitham, Linear and nonlinear waves, Wiley NY, 1974 636 pp.